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One-dimensional pattern formation with Galilean invariance near a stationary bifurcation

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One-dimensional pattern formation with Galilean symmetry is not governed by the Ginzburg-Landau equation near onset; an additional equation describing a large-scale mean flow is required. We derive these amplitude equations and predict the solution amplitude. All steady patterns are unstable, with the growth of the instability taking place on a faster time scale than the formation of the pattern itself. Numerical simulations show that chaotic solutions are obtained, whose amplitude is consistent with our theory.

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In this Rapid Communication we consider one-dimensional pattern formation in a system with Galilean invariance. We suppose that this system is characterized by a velocity component $u(x,t)$, and that in addition to the usual space and time shift symmetries it is invariant under the Galilean symmetry

$$x \to x + Vt, \quad u \to u + V. \tag{1}$$

Then the general one-dimensional governing equation that is first-order in time takes the form

$$u_t + u u_x = \mathcal{F}(u_x, u_{xx}, u_{xxx}, \dots), \tag{2}$$

where $\mathcal{F}$ is an arbitrary function. If the system further satisfies the reflection symmetry

$$x \to -x, \quad u \to -u, \tag{3}$$

then the governing equation for u takes the form

$$u_t + u u_x = \alpha_2 u_{xx} + \alpha_4 u_{xxxx} + \cdots + \beta_{12} u_x u_{xx} + \beta_{23} u_{xx} u_{xxx} + \cdots, \tag{4}$$

where $\alpha_2, \alpha_4, \beta_{12}, \beta_{23}, \dots$ are constants. An example of this type is the well known Kuramoto-Sivashinsky equation $u_t + u u_x = -u_{xx} - u_{xxxx}$ [1,2].

The Galilean symmetry (1) prohibits on the right-hand side of Eq. (4) any linear term in u that does not involve x derivatives, which in turn leads to the existence of a slowly evolving large-scale mode. Our aim is to describe the behavior of systems such as Eq. (4) near the onset of pattern formation with a nonzero wave number. The amplitude equations we derive are universal for such systems. (The Kuramoto-Sivashinsky equation does not come into this category since there is no separation of scales between the pattern and the large-scale mode). Besides systems with Galilean symmetry, another application of our work is in pattern-forming systems with a conservation law and the reflection symmetry (3) [3].

As a specific example, we consider the equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^2}{\partial x^2}\left[r u - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u \right] - u \frac{\partial u}{\partial x}, \tag{5}$$

which has been proposed as a model for viscoelastic media [4]. Although Eq. (5) and the corresponding amplitude equations have been studied previously [5–9], some unanswered questions remain. We determine the scalings for the rms value of u and the growth rate of perturbations to a regular periodic state when r is small.

Linearizing Eq. (5) and seeking solutions proportional to $\exp(\lambda t + ikx)$ [4,9,10] yields the dispersion relation

$$\lambda = k^2[r - (1-k^2)^2], \tag{6}$$

which is shown in Fig. 1. Our interest is in the behavior of Eq. (5) near the onset of pattern formation, when r is small. In this case it is convenient to introduce a small parameter ϵ, defined by $r = \epsilon^2$. For $0 < \epsilon \ll 1$, there is a band of unstable wave numbers with width of order ϵ near $k = 1$. Furthermore, for small k the growth rate λ is of order k^2. It is the interaction of these two classes of modes (pattern modes near $k = 1$ and mean-flow modes near $k = 0$) that we seek to describe.

In the weakly nonlinear regime, $\epsilon \ll 1$, stationary solutions of Eq. (5) exist in the form of ''rolls'' given by

$$u \sim \epsilon a_0 \exp(1+\epsilon q) i x + \epsilon^2 u_2 + \text{c.c.}, \tag{7}$$

where the amplitude a_0 can be taken to be real and can be evaluated by standard weakly nonlinear analysis [9]. At third order a solvability condition gives $a_0 = 6(1-4q^2)^{1/2}$.

To analyze the stability of the rolls in Eq. (7), we write

$$u \sim \epsilon [a_0 + a(X,T)] \exp(1+\epsilon q) i x + \text{c.c.} + \epsilon f(X,T), \tag{8}$$

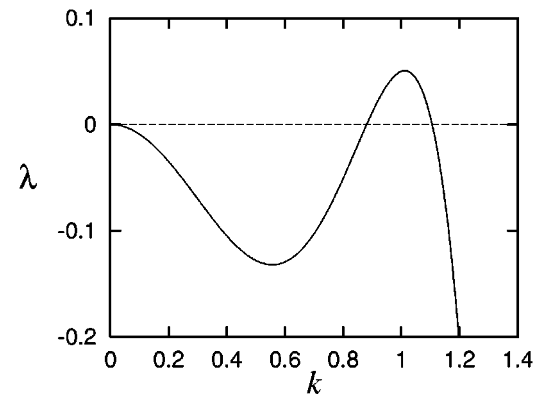

FIG. 1. The growth rate λ of the linear modes of Eq. (5) as a function of wave number k with $r = 0.05$.

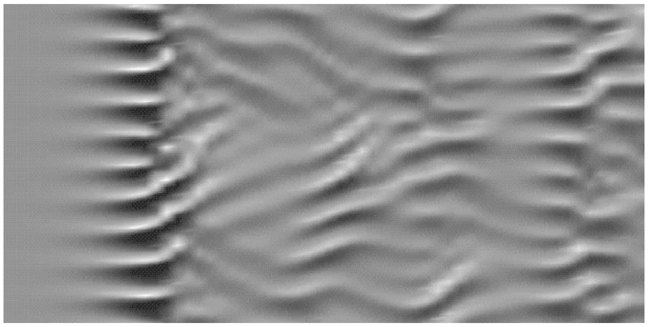

FIG. 2. Grayscale $x-t$ plot of a numerical solution to Eq. (5) with periodic boundary conditions. The width of the domain is 60, the solution is plotted for $0\leq t\leq 300$, the number of grid points is 128, and $r=0.1$.

where X and T are scaled versions of x and t. After writing $a=b+ic$ and linearizing, we find three equations for the perturbations b, c, and f. Since these equations are linear we can arbitrarily choose $b=O(1)$. Then a consistent scaling can be obtained in which

$$f=O(\epsilon^{1/4}),\quad c=O(\epsilon^{-1/4}),\quad X=\epsilon^{3/4}x,\quad T=\epsilon^{3/2}t. \tag{9}$$

With this scaling, the three linearized equations are

$$b_T=4b_{XX}-8qc_X, \tag{10}$$

$$c_T=4c_{XX}-a_0 f, \tag{11}$$

$$f_T=f_{XX}-2a_0 b_X. \tag{12}$$

The stability of this system is determined by seeking solutions proportional to $\exp(\sigma T+ilX)$, yielding the following cubic for the growth rate σ:

$$\sigma^3+9l^2\sigma^2+24l^4\sigma+16l^2(l^4-qa_0^2)=0. \tag{13}$$

For small l, the growth rate obeys $\sigma\sim(16l^2qa_0^2)^{1/3}$, indicating that there is a real positive eigenvalue for $q>0$ and a complex conjugate pair of eigenvalues with positive real part for $q<0$. For $q>0$, there is a stationary bifurcation when the perturbation wave number l obeys $l=l_c=(qa_0^2)^{1/4}$, so the pattern is unstable to a band of wave numbers $0<l<l_c$. For $q<0$ the instability is oscillatory and a Hopf bifurcation occurs when $l=l_o=(-2qa_0^2/25)^{1/4}$, so again there is a band of unstable wave numbers, $0<l<l_o$.

Thus Eq. (5) has the property that all stationary, spatially periodic, small-amplitude patterns are unstable. Furthermore, the instability of the pattern grows at a rate of order $\epsilon^{3/2}$, which is more rapid than the $O(\epsilon^2)$ growth rate of the original pattern. It is also of interest that the perturbation wave number l is of order $\epsilon^{3/4}$, so the destabilizing mode lies outside the $O(\epsilon)$ band of wave numbers where stationary solutions exist.

Given that all patterns are unstable, the question of the small-amplitude behavior of Eq. (5) remains. Figure 2 shows a typical numerical solution of Eq. (5), obtained using a pseudospectral method. The initial condition is a small random perturbation from equilibrium. Initially, a regular pattern forms, with ten waves in the periodic domain. This pattern then abruptly becomes unstable, on a shorter timescale

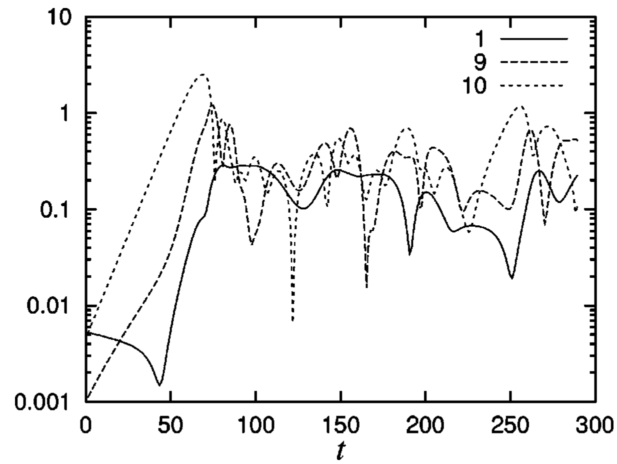

FIG. 3. Mode amplitudes of the first, ninth, and tenth Fourier components for the simulation shown in Fig. 2. The initial slow development of the pattern is followed by a rapid instability and then chaotic behavior at a lower amplitude.

than its evolution, as predicted by the analysis above. The subsequent behavior appears to show spatiotemporal chaos. Evolution of the mode amplitudes is shown in Fig. 3, where the establishment of the pattern can clearly be seen to take place on a time scale ($\epsilon^2 t$) longer than that of the destruction of the pattern ($\epsilon^{3/2}t$).

In attempting to characterize the behavior of a nonlinear partial differential equation such as Eq. (5) near the onset of pattern formation, the usual approach is to reduce it to an amplitude equation. In many mathematical models for pattern formation in one spatial dimension, the Ginzburg-Landau equation [11–13]

$$A_T=A+A_{XX}-|A|^2A \tag{14}$$

is appropriate. In Eq. (14), the complex amplitude A is a function of a large lengthscale X and a long timescale T. The behavior of Eq. (14) is well understood: periodic patterns $A=A_0\exp(iqX)$ are stable provided that the wave number q lies within the Eckhaus band $q^2<1/3$ [14]. However, there are many instances [15] where the onset of pattern formation is not governed by Eq. (14). In the presence of Galilean invariance, an amplitude equation must be included for a large-scale mode [5]. Previous work [8,10] has reached a variety of conclusions about the small-r scaling of the solutions, which we attempt to resolve below.

We now consider how Eq. (14) may appropriately be extended to capture the instability of periodic patterns and the chaotic behavior of Eq. (5). The natural approach, following the standard methods used to obtain Eq. (14), is to write

$$u\sim\epsilon A(X,T)\exp(ix)+\text{c.c.}+\epsilon f(X,T), \tag{15}$$

where $X=\epsilon x$ and $T=\epsilon^2 t$. However, this leads to the following inconsistent amplitude equations:

$$A_T=A+4A_{XX}-|A|^2A/36-ifA/\epsilon-(fA)_X, \tag{16}$$

$$f_T=f_{XX}-|A|^2_X-ff_X, \tag{17}$$

incomplete versions of which have been given previously [5,6,10]. The appearance of the factor ϵ^{-1} in Eq. (16) indicates that the scaling (15) is incorrect. However, if the mean mode f is rescaled to make Eq. (16) consistent then a factor ϵ^{-1} instead multiplies the coupling term in Eq. (17) (see [6,16,17]).

A self-consistent scaling is obtained as follows. Since the growth rate in Eq. (6) is of order ϵ^2, we must retain the scaling $T=\epsilon^2 t$. If the diffusion terms are to appear in the amplitude equations, then $X=\epsilon x$. The leading coupling term in the A equation appears at the correct order if the large-scale mode is of order ϵ^2. Finally, the coupling term in the f equation balances the other terms if the A-mode is of order $\epsilon^{3/2}$. The correct ansatz to replace Eq. (15) is therefore

$$u \sim \epsilon^{3/2} A(X,T)\exp(ix) + \text{c.c.} + \epsilon^2 f(X,T). \tag{18}$$

The corresponding asymptotically consistent amplitude equations are then

$$A_T = A + 4A_{XX} - ifA, \tag{19}$$

$$f_T = f_{XX} - |A|^2_X. \tag{20}$$

Note that the stabilizing cubic term does not appear in Eq. (19) at leading order. This means that Eq. (19) allows exponentially growing solutions in which $A(X,T)=A_0 \exp T$ and $f(X,T)=0$. However, it can be shown that these growing solutions are unstable, in the following sense. By writing $A(X,T)=A_0 \exp T + b(X,T) + ic(X,T)$, linearizing in b, c, and f and seeking solutions with wave number l, we find

$$b_T = (1-4l^2)b, \tag{21}$$

$$c_T = (1-4l^2)c - fA_0 \exp T, \tag{22}$$

$$f_T = -l^2 f - 2ilbA_0 \exp T. \tag{23}$$

The solution for b is thus $b=b_0 \exp((1-4l^2)T)$ and so f includes a term growing as $\exp[(2-4l^2)T]$, which is faster than the growth of the basic state, for small l. Thus, the X-independent solution $A_0 \exp T$ rapidly becomes overtaken by X-dependent perturbations. Numerical simulations of Eqs. (19) and (20) confirm that, in practice, solutions do not grow exponentially in time like $\exp T$, but remain bounded, and that the dynamics is very similar to that of the original equation (5).

According to the new scaling (18), the amplitude of solutions to Eq. (5) should scale as the 3/4 power of the forcing parameter r. In order to check the range of validity of this asymptotic scaling, numerical simulations of Eq. (5) were carried out over a range of values of r in a periodic domain large enough to contain 25 waves. The rms value of the solution was averaged over the time of the simulation. The results in Fig. 4 show an excellent agreement with the 3/4 scaling law over the range $0.01<r<0.1$. For $r>0.1$, the growth rate given by Eq. (6) is of a similar order over the range $0<k<1$ and so the asymptotic scaling (18) no longer holds. Earlier numerical results [8,10] suggested different scalings (with the rms value of u proportional to r [10] or $r^{1/2}$ [8]).

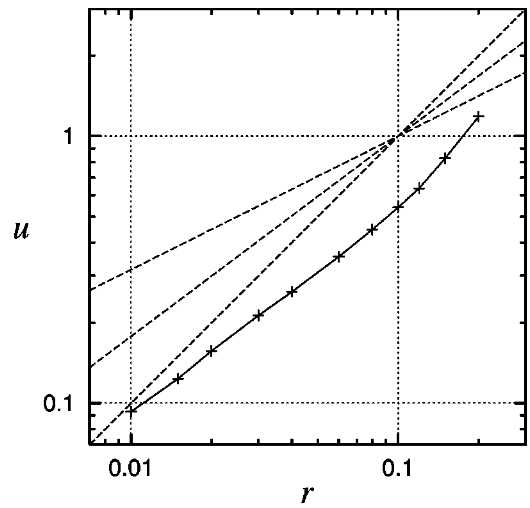

FIG. 4. Log-log plot of rms value of u averaged over 35 000 time units, from numerical simulations of Eq. (5) with 256 grid points and a domain size of 157. The dashed lines have slopes of 1/2, 3/4, and 1 for comparison.

The analysis so far has concentrated on the particular pattern-forming equation (5). However, our results are more widely applicable. For a general equation that is first order in time and satisfies the symmetries (1) and (3), the general form of the amplitude equations (19) and (20) can be written down. Note that the action of the symmetry (1) is

$$f \to f + V, \quad A \to A \exp -iVT, \tag{24}$$

and that Eq. (3) becomes

$$X \to -X, \quad A \to -A^*, \quad f \to -f. \tag{25}$$

The general form of the asymptotic amplitude equations is therefore

$$A_T = A + A_{XX} - ifA, \tag{26}$$

$$f_T = \nu f_{XX} - |A|^2_X. \tag{27}$$

All other terms permitted by the symmetries are of higher order. Note that there is only one free parameter, ν, in the system, since we can rescale X, T, A, and f.

In conclusion, we have examined one-dimensional systems with Galilean symmetry and a reflection symmetry, near a stationary bifurcation with nonzero wave number. We have shown that, in general, a long-wavelength mode destabilizes the pattern near onset. We have derived asymptotically consistent scalings for the pattern near onset and amplitude equations that capture the dynamics.

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